

# On equitable social welfare functions satisfying the Weak Pareto Axiom: A complete characterization

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The present paper examines the problem of aggregating infinite utility streams with a social welfare function that respects the Anonymity and Weak Pareto Axioms. The paper provides a complete characterization of domains (of the one period utilities) on which such an aggregation is possible. A social welfare function satisfying the Anonymity and Weak Pareto Axioms exists on precisely those domains that do not contain any set of the *order type* of the set of positive and negative integers. The criterion is applied to decide on possibility and impossibility results for a variety of domains. It is also used to provide an alternative formulation of the characterization result in terms of the accumulation points of the domain.

**Key words** anonymity axiom, Weak Pareto Axiom, social welfare function, domain restrictions, order types, accumulation points

**JEL classification** D60, D70, D90

Accepted 6 December 2010

## 1 Introduction

Social decision-making in any year often has implications for the welfare of members of society who will live in the future, in addition to those who are living at present. When such decision-making involves sacrificing the well-being of future generations relative to the present (or of the present relative to the future) one must face up to the important question of how to treat the well-being of future generations relative to the well-being of those living at present. This is the subject matter of “intergenerational equity,” which has received considerable attention in economics and philosophy.

The more recent published literature on this topic has been motivated by several real-world intergenerational conflicts, which have required urgent attention at the national and international levels of social decision-making. These include the issues of climate change (abating greenhouse gas emissions to reduce global warming), environmental preservation (exploiting natural resources judiciously and preserving biodiversity) and sustainable

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† We would like to thank Kaushik Basu, David Easley and Kuntal Banerjee for helpful discussions.

development (bequeathing future generations with a larger productive capacity through current capital accumulation).<sup>1</sup>

In his discussion of the concept of intergenerational equity, Ramsey (1928, p. 543) maintains that discounting one generation's utility relative to another's is "ethically indefensible," and something that "arises merely from the weakness of the imagination." Nevertheless, when evaluating long-term policies, economists usually adopt a social welfare function that discounts the welfare of future generations relative to the present. Application of such a discounted utilitarian social welfare function in deciding on the current and future use of non-renewable resources is seen to undermine the well-being of generations far into the future for every positive discount rate, even when sustainable streams with non-decreasing well-being are feasible. Such implications have led several scholars to question the appropriateness of the discounted utilitarian criterion.<sup>2</sup>

Therefore, there is a compelling reason for an "equal treatment" of all generations (present and future) in social decision-making. In the literature on intertemporal social choice, this is formalized in the form of an *Anonymity Axiom* on social preferences, which requires that society should be indifferent between two streams of well-being, if one is obtained from the other by interchanging the levels of well-being of any two generations.

An axiom on social preferences on which there is broad agreement among economists is that it should respect the *Pareto Axiom*. Society should consider one stream of well-being to be superior to another if at least one generation is better off and no generation is worse off in the former compared to the latter.

In the context of a society where the concern for generations extends over an infinite future, we are led to the question of evaluating infinite utility streams consistently with social preferences that respect the Anonymity and Pareto Axioms. This question has received considerable attention in the literature on intertemporal welfare economics, and we provide a brief overview. For this purpose, we use the framework that has become standard in this literature. We consider the problem of defining social welfare orders on the set  $X$  of infinite utility streams, where this set takes the form of  $X = Y^{\mathbb{N}}$ , with  $Y$  denoting a non-empty set of real numbers and  $\mathbb{N}$  the set of natural numbers.

In a seminal contribution, Diamond (1965) shows that there does not exist any continuous social welfare order satisfying the Anonymity and Pareto Axioms (where continuity is defined with respect to the sup metric), when  $Y$  is the closed interval  $[0, 1]$ . A social welfare order satisfying the Pareto Axiom and the continuity requirement is representable by a social welfare function, which is continuous in the sup metric, when  $Y$  is the closed interval  $[0, 1]$ . Therefore, Diamond's result also implies that there does not exist any social welfare function satisfying the Anonymity and Pareto Axioms, which is continuous in the sup metric, when  $Y$  is the closed interval  $[0, 1]$ .

Basu and Mitra (2003) show that this last statement can be refined as follows: there does not exist any social welfare function satisfying the Anonymity and Pareto Axioms, when  $Y$  contains at least two distinct elements. Another way of stating this is that there does not exist

<sup>1</sup> See the recent comprehensive survey on intergenerational equity by Asheim (2010).

<sup>2</sup> For elaboration of this point, see the discussions in Asheim (2010), and Asheim, Mitra, and Tungodden (2010), who also provide references to the relevant published literature on this issue.

any *representable* social welfare order satisfying the Anonymity and Pareto Axioms, when  $Y$  contains at least two distinct elements. That is, by directly assuming representability of the social welfare order, one can dispense with the continuity requirement in Diamond's result. Perhaps more important from the point of view of the present investigation, is that the domain restriction is minimal, because there is no social decision problem when  $Y$  has fewer than two elements.

If one requires neither continuity of the social welfare order nor its representability, it is possible to show the existence of a social welfare order satisfying the Anonymity and Pareto Axioms. Svensson (1980) establishes this important result, assuming  $Y$  to be the closed interval  $[0, 1]$ . However, his possibility result uses Szpilrajn's Lemma, and so the social welfare order is non-constructive; it cannot be used by policy-makers for social decision-making. Zame (2007) and Lauwers (2010) show that it is not possible to obtain a social welfare order satisfying the Anonymity and Pareto Axioms without resort to some non-constructive device.

This brief review indicates that there is a genuine conflict between intergenerational equity and efficiency criteria in the evaluation of infinite utility streams.<sup>3</sup> However, Basu and Mitra (2007b) show that with domain restrictions, it is possible to have social welfare functions on infinite utility streams that satisfy simultaneously the Anonymity Axiom and a weaker form of the Pareto Axiom.

With respect to domain restrictions, they argue that in reality the possible values that each generation's utility can take may be quite limited. While it might be mathematically convenient to assume that the possible values that a generation's utility can take can be represented by the set of reals (or the set of reals in  $[0, 1]$ ), it might be more realistic to suppose that individual utilities can take a finite number of values or, at most, a countably infinite number of values. Of course, domain restrictions by themselves will not be able to prevent the conflict between equity and efficiency, given the general impossibility theorem of Basu and Mitra (2003), which applies (as already noted above) to all non-trivial domains.

With respect to weakening the Pareto Axiom, one can justifiably take the position that the so-called *Weak Pareto Axiom* is more compelling than the Pareto Axiom; it requires that society should consider one stream of well-being to be superior to another if *every* generation is better off in the former compared to the latter. In the context of evaluating infinite utility streams, it is debatable whether in comparing two utility streams society is always better off if one generation is (or a finite number of generations are) better off and all other generations are unaffected, so the standard Pareto Axiom might not be self-evident.

If we accept the Anonymity Axiom and the Weak Pareto Axiom as our guiding principles of intergenerational equity and efficiency (respectively), the contribution of Basu and Mitra (2007b) raises the question as to what exactly is the nature of the domain restrictions that allows their possibility result to emerge. The present paper is devoted to providing a systematic and complete answer to this question.

<sup>3</sup> Our review does not cover many papers on this topic. For a sample of the literature, see Koopmans (1972), Chichilnisky (1996), Lauwers (1998), Shinotsuka (1998), Basu and Mitra (2007a, b), Fleurbaey and Michel (2003), Asheim and Tungodden (2004), Banerjee (2006), Sakai (2006), Asheim, Mitra, and Tungodden (2007), Bossert, Sprumont, and Suzumura (2007), Hara, Shinotsuka, Suzumura, and Xu (2008) and Crespo, Núñez, and Rincón-Zapatero (2009).

We consider the problem of defining social welfare functions on the set  $X$  of infinite utility streams, where this set takes the form of  $X = Y^{\mathbb{N}}$ , with  $Y$  denoting a non-empty subset of the reals and  $\mathbb{N}$  the set of natural numbers. In discussing “domain restrictions” we refer to the set  $Y$  as the “domain” as a short-hand, even though the domain of the social welfare function is actually  $X$ . This is because we would like to study the nature of  $Y$  that allows for possibility results, and would like to give easily verifiable conditions on  $Y$  that can be checked, instead of conditions on the set  $X$ , which might be considerably harder to verify.

When  $Y = \mathbb{N}$ , Basu and Mitra (2007b, Theorem 3) show that there is a social welfare function on  $X = Y^{\mathbb{N}}$  that satisfies the Anonymity and Weak Pareto Axioms. In fact, one can write this function explicitly as the “min” function, noting that the minimum for every infinite utility stream in  $X$  exists when  $Y = \mathbb{N}$ . In contrast, if  $Y = [0, 1]$ , Basu and Mitra (2007b, Theorem 4) show that there is no social welfare function on  $X = Y^{\mathbb{N}}$ , which satisfies the Anonymity and Weak Pareto Axioms.

This leads to the question of whether it is the countability of the set  $Y$  that is crucial in allowing possibility results to emerge. This turns out not to be the case: we show that when  $Y = \mathbb{I}$ , where  $\mathbb{I}$  is the set of positive and negative integers, there is no social welfare function on  $X = Y^{\mathbb{N}}$  that satisfies the Anonymity and Weak Pareto Axioms. In fact, we go further and provide a complete characterization of the domains,  $Y$ , for which there exists a social welfare function satisfying the Anonymity and Weak Pareto Axioms. These are precisely those domains that do not contain any set of the *order type* of the set of positive and negative integers.<sup>4</sup>

Our characterization result provides a new perspective on known results in the literature as well as new results for domains for which the existing literature has little to offer. For instance, an easy implication of our characterization result is that if  $Y$  is the set of rationals in  $[0, 1]$ , then there is no social welfare function on  $X$  respecting the Anonymity and Weak Pareto Axioms.<sup>5</sup> Not only is this result new in the literature, but the currently available methods provide no hint as to how an impossibility result for this domain might even be approached.

The possibility part of the result is especially useful because the social welfare function can be written in explicit form with a formula involving a weighted average of the sup and inf functions on  $X$  (using simple monotone transformations of the elements of  $X$ , if needed). Therefore, whenever there is a social welfare function respecting the Anonymity and Weak Pareto Axioms, this particularly simple form will suffice.

The characterization is given in terms of order types. This criterion is applicable for determining possibility and impossibility results for a variety of domains. This is demonstrated by presenting a number of illustrative examples.

The criterion is also used to provide a reformulation of the characterization result in terms of *right* and *left accumulation points* of the domain. This alternative characterization is shown to be easier to apply to domains to determine possibility and impossibility results.

<sup>4</sup> The term “order type” is explained in Section 2.

<sup>5</sup> This result is of interest since a compelling case can surely be made to restrict the possible values of a generation’s utility to the rationals.

## 2 Formal setting and main result

### 2.1 Weak Pareto and Anonymity Axioms

Let  $\mathbb{R}$  be the set of real numbers,  $\mathbb{N}$  the set of positive integers, and  $\mathbb{I}$  the set of positive and negative integers. Suppose  $Y \subset \mathbb{R}$  is the set of all possible utilities that any generation can achieve. Then  $X = Y^{\mathbb{N}}$  is the set of all possible utility streams. If  $\langle x_n \rangle \in X$ , then  $\langle x_n \rangle = (x_1, x_2, \dots)$ , where, for all  $n \in \mathbb{N}$ ,  $x_n \in Y$  represents the amount of utility that the generation of period  $n$  earns. For all  $y, z \in X$ , we write  $y \geq z$  if  $y_n \geq z_n$  for all  $n \in \mathbb{N}$ ; we write  $y > z$  if  $y \geq z$  and  $y \neq z$ ; and we write  $y \gg z$  if  $y_n > z_n$  for all  $n \in \mathbb{N}$ .

If  $Y$  has only one element, then  $X$  is a singleton, and the problem of ranking or evaluating infinite utility streams is trivial. Therefore, the set  $Y$  will always be assumed to have at least two distinct elements.

A social welfare function (SWF) is a mapping  $W : X \rightarrow \mathbb{R}$ . Consider now the axioms that we may want the SWF to satisfy. The first axiom is the Weak Pareto condition; this is a version of the Pareto Axiom that has been widely used in the literature (see Arrow 1963; Sen 1977), and is probably even more compelling than the standard Pareto Axiom.<sup>6</sup>

**Weak Pareto Axiom:** For all  $x, y \in X$ , if  $x \gg y$ , then  $W(x) > W(y)$ .

The next axiom is the one that captures the notion of “inter-generational equity”; we shall call it the “Anonymity Axiom.”<sup>7</sup>

**Anonymity Axiom:** For all  $x, y \in X$ , if there exist  $i, j \in \mathbb{N}$  such that  $x_i = y_j$  and  $x_j = y_i$ , and for every  $k \in \mathbb{N} \sim \{i, j\}$ ,  $x_k = y_k$ , then  $W(x) = W(y)$ .<sup>8</sup>

### 2.2 Domain types

In this subsection, we recall a few concepts from the mathematical literature dealing with *types* of spaces, which are *strictly ordered* by a binary relation.

We will say that the set  $A$  is strictly ordered by a binary relation  $R$  if  $R$  is *connected* (if  $a, a' \in A$  and  $a \neq a'$ , then either  $aRa'$  or  $a'Ra$  holds), *transitive* (if  $a, a', a'' \in A$  and  $aRa'$  and  $a'Ra''$  hold, then  $aRa''$  holds) and *irreflexive* ( $aRa$  holds for no  $a \in A$ ). In this case, the strictly ordered set will be denoted by  $A(R)$ . For example, the set  $\mathbb{N}$  is strictly ordered by the binary relation  $<$  (where  $<$  denotes the usual “less than” relation on the reals).

We will say that a strictly ordered set  $A'(R')$  is *similar* to the strictly ordered set  $A(R)$  if there is a one-to-one function  $f$  mapping  $A$  onto  $A'$ , such that:

$$a_1, a_2 \in A \text{ and } a_1 R a_2 \implies f(a_1) R' f(a_2).$$

<sup>6</sup> The standard Pareto Axiom is:

**Pareto Axiom:** For all  $x, y \in X$ , if  $x > y$ , then  $W(x) > W(y)$ . We caution the reader that in some of the literature, what we are calling “Weak Pareto” is often called “Pareto,” with the suffix “strong” added to what we have called the “Pareto Axiom.”

<sup>7</sup> In informal discussions throughout the paper, the terms “equity” and “anonymity” are used interchangeably.

<sup>8</sup> If  $A$  and  $B$  are two subsets of  $S$ , the *difference*  $B \sim A$  is the set  $\{z: z \in B \text{ and } z \notin A\}$ . This notation follows Royden (1988).

We now specialize to strictly ordered subsets of the reals. With  $Y$  a non-empty subset of  $\mathbb{R}$ , let us define some order types as follows.<sup>9</sup> We will say that the strictly ordered set  $Y(<)$  is:

- (i) of order type  $\omega$  if  $Y(<)$  is similar to  $\mathbb{N}(<)$ ;
- (ii) of order type  $\sigma$  if  $Y(<)$  is similar to  $\mathbb{I}(<)$ ; and
- (iii) of order type  $\mu$  if  $Y$  contains a non-empty subset  $Y'$ , such that the strictly ordered set  $Y'(<)$  is of order type  $\sigma$ .

The characterization of these types of strictly ordered sets is facilitated by the concepts of a cut, a first element and a last element of a strictly ordered set.

Given a strictly ordered set  $Y(<)$ , let us define a *cut*  $[Y_1, Y_2]$  of  $Y(<)$  as a partition of  $Y$  into two non-empty sets  $Y_1$  and  $Y_2$  (i.e.  $Y_1$  and  $Y_2$  are non-empty,  $Y_1 \cup Y_2 = Y$  and  $Y_1 \cap Y_2 = \emptyset$ ), such that for each  $y_1 \in Y_1$  and each  $y_2 \in Y_2$ , we have  $y_1 < y_2$ .

An element  $y_0 \in Y$  is called a *first element* of  $Y(<)$  if  $y < y_0$  holds for no  $y \in Y$ . An element  $y^0 \in Y$  is called a *last element* of  $Y(<)$  if  $y^0 < y$  holds for no  $y \in Y$ .

The following result can be found in Sierpinski (1965).

**Lemma 1** *A strictly ordered set  $Y(<)$  is of order type  $\sigma$  if and only if the following two conditions hold:*

- (i)  $Y$  has neither a first element nor a last element.
- (ii) For every cut  $[Y_1, Y_2]$  of  $Y$ , the set  $Y_1$  has a last element and the set  $Y_2$  has a first element.

### 2.3 The characterization result

The complete characterization result of the present paper can now be stated as follows.

**Theorem 1** *Let  $Y$  be a non-empty subset of  $\mathbb{R}$ . There exists a social welfare function  $W : X \rightarrow \mathbb{R}$  (where  $X = Y^{\mathbb{N}}$ ) satisfying the Weak Pareto and Anonymity Axioms if and only if  $Y(<)$  is not of order type  $\mu$ .*

The result implies that there is a social welfare function  $W : X \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms (where  $X = Y^{\mathbb{N}}$ ), when  $Y = \mathbb{N}$ , but that there is no such function when  $Y = \mathbb{I}$ . Additional examples show that the criterion given is easy to check to determine possibility and impossibility results.

<sup>9</sup> The name “order type  $\omega$ ” appears in Sierpinski (1965). The name “order type  $\sigma$ ” is our own, but it is discussed and characterized in Sierpinski (1965). The name “order type  $\mu$ ” is our own, and it appears to be the crucial concept for the problem we are studying.

### 3 The possibility result

We first present the possibility part of the result in Theorem 1 for domains  $Y \subset [0, 1]$ . This enables one to explicitly define a social welfare function, and verify that, when the domain  $Y$  is such that  $Y(<)$  is not of type  $\mu$ , the function satisfies the Weak Pareto and Anonymity Axioms.

The explicit form of the social welfare function makes this possibility result potentially useful for policy purposes. In addition, the social welfare function has the desirable property that it satisfies the following “monotonicity condition”:<sup>10</sup>

$$\text{For all } x, x' \in X, \text{ if } x > x', \text{ then } W(x) \geq W(x'). \tag{M}$$

**Proposition 1** *Let  $Y$  be a non-empty subset of  $[0, 1]$  and suppose that  $Y(<)$  is not of order type  $\mu$ . For  $x = (x_n)_{n=1}^\infty \in X \equiv Y^\mathbb{N}$ , define:*

$$W(x) = \alpha \inf \{x_n\}_{n \in \mathbb{N}} + (1 - \alpha) \sup \{x_n\}_{n \in \mathbb{N}},$$

where  $\alpha \in (0, 1)$  is a parameter. Then  $W$  satisfies the Anonymity and Weak Pareto Axioms.

**PROOF: (Anonymity)** For any  $x \in X$ ,  $W(x)$  depends only on the set  $\{x_n\}_{n \in \mathbb{N}}$ . This set does not change with any finite permutation of its elements. Hence,  $W(x)$  also does not change with any such permutation. Therefore,  $W$  satisfies the Anonymity Axiom.

**(Weak Pareto)** Let  $x, x' \in X$  with  $x' \gg x$ . We claim that  $W(x') > W(x)$ . Clearly, by definition of  $W$ , we have  $W(x') \geq W(x)$  and, therefore, if the claim is false, it must be the case that:

$$W(x') = W(x). \tag{1}$$

Because:

$$\inf \{x'_n\}_{n \in \mathbb{N}} \geq \inf \{x_n\}_{n \in \mathbb{N}} \text{ and } \sup \{x'_n\}_{n \in \mathbb{N}} \geq \sup \{x_n\}_{n \in \mathbb{N}} \tag{2}$$

and  $\alpha \in (0, 1)$ , it follows that:

$$a \equiv \inf \{x'_n\}_{n \in \mathbb{N}} = \inf \{x_n\}_{n \in \mathbb{N}} \text{ and } b \equiv \sup \{x'_n\}_{n \in \mathbb{N}} = \sup \{x_n\}_{n \in \mathbb{N}}. \tag{3}$$

Clearly,  $a, b \in [0, 1]$  and  $b \geq a$ . In fact, we must have  $b > a$ , because:

$$b \equiv \sup \{x'_n\}_{n \in \mathbb{N}} \geq x'_1 > x_1 \geq \inf \{x'_n\}_{n \in \mathbb{N}} \equiv a. \tag{4}$$

We now break up our analysis into the following cases:

- (i)  $\{x'_n\}_{n \in \mathbb{N}}$  has a minimum.
- (ii)  $\{x'_n\}_{n \in \mathbb{N}}$  does not have a minimum.

<sup>10</sup> A weak version of Pareto, which requires that the “monotonicity condition” (M), together with what we have called Weak Pareto Axiom, be satisfied, is quite appealing, and has been proposed and examined by Diamond (1965).

Case (ii) is further subdivided as follows:

- (ii) (a)  $\{x'_n\}_{n \in \mathbb{N}}$  does not have a minimum and  $\{x_n\}_{n \in \mathbb{N}}$  has a maximum.
- (ii) (b)  $\{x'_n\}_{n \in \mathbb{N}}$  does not have a minimum and  $\{x_n\}_{n \in \mathbb{N}}$  does not have a maximum.

In case (i), let  $k \in \mathbb{N}$  be such that  $x'_k = \min\{x'_n\}_{n \in \mathbb{N}}$ . Then, we have:

$$a \equiv \inf\{x'_n\}_{n \in \mathbb{N}} = \min\{x'_n\}_{n \in \mathbb{N}} = x'_k > x_k \geq \inf\{x_n\}_{n \in \mathbb{N}} \equiv a,$$

a contradiction.

In case (ii) (a), let  $s \in \mathbb{N}$  be such that  $x_s = \max\{x_n\}_{n \in \mathbb{N}}$ . Then, we have:

$$b \equiv \sup\{x_n\}_{n \in \mathbb{N}} = \max\{x_n\}_{n \in \mathbb{N}} = x_s < x'_s \leq \sup\{x'_n\}_{n \in \mathbb{N}} = b,$$

a contradiction.

Finally, we turn to case (ii) (b). Choose  $c \in (a, b)$ . Then, we can find  $c < x_{n_1} < x_{n_2} < x_{n_3} < \dots$  with  $x_{n_k} \in (c, b)$  for  $k = 1, 2, 3, \dots$ , and  $x_{n_k} \uparrow b$  as  $k \uparrow \infty$ . Similarly, we can find  $x'_{m_1} > x'_{m_2} > x'_{m_3} > \dots$  with  $x'_{m_r} \in (a, c)$  for  $r = 1, 2, 3, \dots$ , and  $x'_{m_r} \downarrow a$  as  $r \uparrow \infty$ . That is, we have:

$$a < \dots < x'_{m_3} < x'_{m_2} < x'_{m_1} < c < x_{n_1} < x_{n_2} < x_{n_3} < \dots < b. \tag{5}$$

Consider the set  $Y' = \{x_{n_1}, x_{n_2}, x_{n_3}, \dots\} \cup \{x'_{m_1}, x'_{m_2}, x'_{m_3}, \dots\}$ . Clearly,  $Y'$  is a subset of  $Y$  and because of (5), we note that (A)  $Y'$  has neither a maximum nor a minimum, and (B) for every cut  $[Y'_1, Y'_2]$  of  $Y'$ , the set  $Y'_1$  has a last element and the set  $Y'_2$  has a first element. Thus, by Lemma 1,  $Y'(<)$  is of order type  $\sigma$ . This means that  $Y(<)$  is of order type  $\mu$ , a contradiction.

Because we are led to a contradiction in cases (i), (ii)(a) and (ii)(b), and these exhaust all logical possibilities, (1) cannot hold, and our claim that  $W(x') > W(x)$  is established. □

Although the possibility result in Proposition 1 is stated for domains  $Y \subset [0, 1]$ , we will see that the result actually holds for all non-empty domains  $Y \subset \mathbb{R}$  (as claimed in Theorem 1) because of an *invariance* result. This states that any possibility result is invariant with respect to monotone transformations of the domain.

**Proposition 2** *Let  $Y$  be a non-empty subset of  $\mathbb{R}$ ,  $X \equiv I^{\mathbb{N}}$ , and  $W : X \rightarrow \mathbb{R}$  be a function satisfying the Weak Pareto and Anonymity Axioms. Suppose  $f$  is a monotone (increasing or decreasing) function from  $I$  to  $Y$ , where  $I$  is a non-empty subset of  $\mathbb{R}$ . Then, there is a function  $V : J \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms, where  $J = I^{\mathbb{N}}$ .*

PROOF: We treat two cases: (i)  $f$  is increasing; and (ii)  $f$  is decreasing.

- (i) Let  $f$  be an increasing function. Define  $V : J \rightarrow \mathbb{R}$  by:

$$V(z_1, z_2, \dots) = W(f(z_1), f(z_2), \dots). \tag{6}$$

Then,  $V$  is well-defined, because  $f$  maps  $I$  into  $Y$ .



To check that  $V$  satisfies the Anonymity Axiom, let  $z, z' \in J$ , with  $z'_r = z_s, z'_s = z_r$ , and  $z'_i = z_i$  for all  $i \neq r, s$ . Without loss of generality, assume  $r < s$ . Then,

$$\begin{aligned} V(z'_1, z'_2, \dots) &= W(f(z'_1), f(z'_2), \dots, f(z'_r), \dots, f(z'_s), \dots) \\ &= W(f(z'_1), f(z'_2), \dots, f(z'_s), \dots, f(z'_r), \dots) \\ &= W(f(z_1), f(z_2), \dots, f(z_r), \dots, f(z_s), \dots) \\ &= V(z_1, z_2, \dots), \end{aligned} \tag{7}$$

the second line of (7) following from the fact that  $W$  satisfies the Anonymity Axiom on  $X$ . Note that the fact that  $f$  is increasing is used nowhere in this demonstration.

To check that  $V$  satisfies the Weak Pareto Axiom, let  $z, z' \in J$  with  $z' \gg z$ . We have:

$$f(z'_i) = f(z_i) + [f(z'_i) - f(z_i)] = f(z_i) + \varepsilon_i \text{ for each } i \in \mathbb{N}, \tag{8}$$

where  $\varepsilon_i \equiv [f(z'_i) - f(z_i)] > 0$  for each  $i \in \mathbb{N}$ , because  $f$  is increasing. Consequently,

$$\begin{aligned} V(z'_1, z'_2, \dots) &= W(f(z'_1), f(z'_2), \dots) \\ &= W(f(z_1) + \varepsilon_1, f(z_2) + \varepsilon_2, \dots) \\ &> W(f(z_1), f(z_2), \dots) \\ &= V(z_1, z_2, \dots), \end{aligned} \tag{9}$$

where the third line of (9) follows from the facts that  $W$  satisfies the Weak Pareto Axiom on  $X, f(z_i) \in Y, f(z_i) + \varepsilon_i \equiv f(z'_i) \in Y$  for all  $i \in \mathbb{N}$ , and  $\varepsilon_i > 0$  for all  $i \in \mathbb{N}$ .

(ii) Let  $f$  be a decreasing function. Define  $V : J \rightarrow \mathbb{R}$  by:

$$V(z_1, z_2, \dots) = -W(f(z_1), f(z_2), \dots). \tag{10}$$

Then,  $V$  is well-defined, because  $f$  maps  $I$  into  $Y$ .

One can check that  $V$  satisfies the Anonymity Axiom by following the steps used in (i) above. To check that  $V$  satisfies the Weak Pareto Axiom, let  $z, z' \in J$  with  $z' \gg z$ . We have:

$$f(z'_i) = f(z_i) + [f(z'_i) - f(z_i)] = f(z_i) - \varepsilon_i \text{ for each } i \in \mathbb{N}, \tag{11}$$

where  $\varepsilon_i \equiv [f(z_i) - f(z'_i)] > 0$  for each  $i \in \mathbb{N}$ , because  $f$  is decreasing. Consequently,

$$\begin{aligned} -V(z'_1, z'_2, \dots) &= W(f(z'_1), f(z'_2), \dots) \\ &= W(f(z_1) - \varepsilon_1, f(z_2) - \varepsilon_2, \dots) \\ &< W(f(z_1), f(z_2), \dots) \\ &= -V(z_1, z_2, \dots), \end{aligned} \tag{12}$$

where the third line of (12) follows from the facts that  $W$  satisfies the Weak Pareto Axiom on  $X, f(z_i) \in Y, f(z_i) - \varepsilon_i \equiv f(z'_i) \in Y$  for all  $i \in \mathbb{N}$ , and  $\varepsilon_i > 0$  for all  $i \in \mathbb{N}$ .

Therefore, we have:

$$V(z'_1, z'_2, \dots) > V(z_1, z_2, \dots). \tag{13}$$

□

We can now state the possibility result claimed in Theorem 1 as follows.

**Proposition 3** *Let  $Y$  be a non-empty subset of  $\mathbb{R}$ . There exists a social welfare function  $W : X \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms (where  $X \equiv Y^{\mathbb{N}}$ ) if  $Y(<)$  is not of order type  $\mu$ .*

PROOF: Let us define a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  by:

$$f(y) = \left(\frac{1}{2}\right) \left[1 + \frac{y}{1 + |y|}\right] \text{ for all } y \in \mathbb{R}. \tag{14}$$

Clearly,  $f$  is an increasing function from  $\mathbb{R}$  to  $(0, 1)$ .

Denote  $f(Y)$  by  $A$ ; then  $A$  is a non-empty subset of  $(0, 1)$ . We claim that  $A(<)$  is not of order type  $\mu$ . For if  $A(<)$  is of order type  $\mu$ , then there is a non-empty subset  $A'$  of  $A$  such that  $A'(<)$  is of order type  $\sigma$ . Define  $C = \{y \in Y: f(y) \in A'\}$ . Then  $C$  is a non-empty subset of  $Y$  and  $f$  is an increasing function from  $C$  onto  $A'$ . Therefore,  $C(<)$  is similar to  $A'(<)$  and so  $C(<)$  is of order type  $\sigma$ . Clearly,  $C$  is a non-empty subset of  $Y$ , and so  $Y(<)$  must be of order type  $\mu$ , a contradiction. This establishes our claim.

Because  $A$  is a non-empty subset of  $(0, 1)$  and  $A(<)$  is not of order type  $\mu$ , we can apply Proposition 1 to obtain a function  $U : B \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms, where  $B \equiv A^{\mathbb{N}}$ .

Because  $f$  is an increasing function from  $Y$  to  $A$  and  $Y$  is a non-empty subset of  $\mathbb{R}$ , we can now apply Proposition 2 to obtain a function  $W : X \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms, where  $X = Y^{\mathbb{N}}$ . □

We now discuss examples which illustrate the usefulness of Proposition 3.

**Example 1** Let  $Y = \mathbb{N}$  and  $X = Y^{\mathbb{N}}$ . We claim that  $Y(<)$  is not of order type  $\mu$ , for if  $Y(<)$  is of order type  $\mu$ , then  $Y$  contains a non-empty subset  $Y'$  such that  $Y'(<)$  is of order type  $\sigma$ . Therefore, by Lemma 1,  $Y'(<)$  has no first element. However, any non-empty subset of  $\mathbb{N}(<)$  has a first element (Munkres 1975, Theorem 4.1). This contradiction establishes the claim.

Using Proposition 3, there is a function  $W : X \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms, where  $X = Y^{\mathbb{N}}$ . This provides an alternative approach to the possibility result noted in Basu and Mitra (2007b, Theorem 3), and in Lauwers (2010, p. 37).

**Example 2** Let  $Y$  be defined by:

$$Y = \{1/n\}_{n \in \mathbb{N}}$$

and let  $X = Y^{\mathbb{N}}$ . We claim that  $Y(<)$  is not of order type  $\mu$ , for if  $Y(<)$  is of order type  $\mu$ , then  $Y$  contains a non-empty subset  $Y'$  such that  $Y'(<)$  is of order type  $\sigma$ . Then, defining:

$$Z = \{(1/y) : y \in Y'\}$$

we see that  $Z$  is a non-empty subset of  $\mathbb{N}$ . Therefore,  $Z(<)$  has a first element and so  $Y'(<)$  has a last element. But, by Lemma 1,  $Y'(<)$  cannot have a last element. This contradiction establishes the claim.

Using Proposition 3, there is a function  $W : X \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms, where  $X = Y^{\mathbb{N}}$ . This result is mentioned without proof in Basu and Mitra (2007b, footnote 9, p. 83).

**Example 3** Define  $A = \{-1/n\}_{n \in \mathbb{N}}$ ,  $B = \{1/n\}_{n \in \mathbb{N}}$  and  $Y = A \cup B$ ,  $X = Y^{\mathbb{N}}$ . We claim that  $Y(<)$  is not of order type  $\mu$ , for if  $Y(<)$  is of order type  $\mu$ , then  $Y$  contains a non-empty subset  $Y'$  such that  $Y'(<)$  is of order type  $\sigma$ .

Define  $A' = A \cap Y'$  and  $B' = B \cap Y'$ . If  $B'$  is non-empty, then  $B'$  is a non-empty subset of  $B$  and, therefore,  $B'(<)$  has a last element (see Example 2 above), which we call  $b$ . If  $A'$  is empty, then  $Y' = B'$  and  $Y'(<)$  has a last element, contradicting Lemma 1. If  $A'$  is non-empty, then for every  $y \in A'$ , we have  $y < b$ . Therefore,  $b$  is a last element of  $Y'(<)$ , contradicting Lemma 1 again.

If  $B'$  is empty, then  $Y' = A'$ . Furthermore,  $A'$  is a non-empty subset of  $A$  and, therefore, has a first element (see Example 2 above). Thus,  $Y'(<)$  must have a first element, contradicting Lemma 1.

The above cases exhaust all logical possibilities and, therefore, our claim is established. Using Proposition 3, there is a function  $W : X \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms, where  $X = Y^{\mathbb{N}}$ .

#### 4 The impossibility result

We will first present the impossibility part of the result in Theorem 1 for the domain  $Y = \mathbb{I}$ , the set of positive and negative integers. Clearly,  $\mathbb{I}(<)$  is of type  $\sigma$  and, therefore, of type  $\mu$ . This enables us to illustrate our approach to the impossibility result in the most transparent way. We will then use Proposition 2 to show that when an arbitrary non-empty subset,  $Y$ , of the reals is such that  $Y(<)$  is of type  $\mu$ , there is no social welfare function satisfying the Weak Pareto and Anonymity Axioms.

**Proposition 4** *Let  $Y = \mathbb{I}$ . Then there is no social welfare function  $W : X \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms (where  $X = Y^{\mathbb{N}}$ ).*

PROOF: Suppose on the contrary that there is a social welfare function  $W : X \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms (where  $X \equiv Y^{\mathbb{N}} = \mathbb{I}^{\mathbb{N}}$ ).

Let  $Q$  be a fixed enumeration of the rationals in  $(0, 1)$ . Then, we can write:

$$Q = \{q_1, q_2, q_3, \dots\}.$$

For any real number  $t \in (0, 1)$ , there are infinitely many rational numbers from  $Q$  in  $(0, t)$  and in  $[t, 1)$ .

For each real number  $t \in (0, 1)$ , we can then define the set  $M(t) = \{n \in \mathbb{N} : q_n \in (0, t)\}$  and the sequence  $\langle m_s(t) \rangle$  as follows:

$$m_1(t) = \min\{n \in \mathbb{N} : q_n \in (0, t)\}.$$

For  $s \in \mathbb{N}, s > 1$ ,

$$m_s(t) = \min\{n \in \mathbb{N} \sim \{m_1(t), \dots, m_{s-1}(t)\} : q_n \in (0, t)\}.$$

The sequence  $\langle m_s(t) \rangle$  is then well-defined, and:

$$m_1(t) < m_2(t) < m_3(t) \dots$$

and  $M(t) = \{m_1(t), m_2(t), \dots\}$ .

For each real number  $t \in (0, 1)$ , we can define the set  $P(t) = \{n \in \mathbb{N} : q_n \in [t, 1)\}$  and the sequence  $\langle p_r(t) \rangle$  as follows:

$$p_1(t) = \min\{n \in \mathbb{N} : q_n \in [t, 1)\}.$$

For  $r \in \mathbb{N}, r > 1$ ,

$$p_r(t) = \min\{n \in \mathbb{N} \sim \{p_1(t), \dots, p_{r-1}(t)\} : q_n \in [t, 1)\}.$$

The sequence  $\langle p_r(t) \rangle$  is then well-defined, and:

$$p_1(t) < p_2(t) < p_3(t) \dots$$

and  $P(t) = \{p_1(t), p_2(t), \dots\}$ .

To make the exposition transparent, we now break up the proof into four steps.

**Step 1** (Defining the sequence  $\langle x(t) \rangle$ )

For each real number  $t \in (0, 1)$ , we note that  $M(t) \cap P(t) = \emptyset$ , and  $M(t) \cup P(t) = \mathbb{N}$ . Then, we can define a sequence  $\langle x(t) \rangle$  as follows:

$$x_n(t) = \begin{cases} 2s - 1 & \text{if } n = m_s \text{ for some } s \in \mathbb{N} \\ -2r - 1 & \text{if } n = p_r \text{ for some } r \in \mathbb{N} \end{cases} \tag{15}$$

Note that the sequence  $\langle x_n(t) \rangle$  will contain, by (15), all the positive odd numbers in increasing order of magnitude with  $n$ , and all the negative odd numbers less than  $(-1)$  in decreasing order of magnitude with  $n$ .

**Step 2** (Comparing  $\langle x(\alpha) \rangle$  with  $\langle x(\beta) \rangle$ )

Let  $\alpha, \beta$  be arbitrary real numbers in  $(0, 1)$ , with  $\alpha < \beta$ . Note that if  $n \in M(\alpha)$ , then  $n \in M(\beta)$ , and if  $n \in P(\beta)$  then  $n \in P(\alpha)$ . Because there are an infinite number of rationals

from  $Q$  in  $[\alpha, \beta)$ , there will be an infinite number of distinct elements of  $\mathbb{N}$  in:

$$L(\alpha, \beta) = M(\beta) \cap P(\alpha) = \{n \in \mathbb{N} : q_n \in [\alpha, \beta)\}.$$

For any  $n \in L(\alpha, \beta)$ , we have  $n \in M(\beta)$  but  $n \notin M(\alpha)$ . That is, by (15), for each  $n \in L(\alpha, \beta)$  it must be the case that  $x_n(\alpha) < 0$  but  $x_n(\beta) > 0$ . Consequently, one has:

$$x_n(\beta) \geq x_n(\alpha) \text{ for all } n \in \mathbb{N}. \tag{16}$$

Informally, these observations may be expressed as follows. In comparing the sequence  $\langle x(\alpha) \rangle$  with  $\langle x(\beta) \rangle$ , whenever  $\langle x(\alpha) \rangle$  has a positive entry for some coordinate, there must be a positive entry for that coordinate in  $\langle x(\beta) \rangle$ . There will be an infinite number of coordinates (switches) for which  $\langle x(\alpha) \rangle$  will have a negative entry, but for which  $\langle x(\beta) \rangle$  will have a positive entry. For the remaining coordinates, both  $\langle x(\alpha) \rangle$  and  $\langle x(\beta) \rangle$  will have negative entries. Because of the switches,  $\langle x(\beta) \rangle$  uses up the sub-indices in  $M(\beta)$  earlier and postpones using the sub-indices in  $P(\beta)$  until later compared to  $\langle x(\alpha) \rangle$ , leading to (16).

One can strengthen the conclusion in (16) as follows. This also formalizes the informal observations given above. Define:

$$N = \min\{n \in \mathbb{N} : n \in L(\alpha, \beta)\}.$$

Then, by (15), we have  $x_N(\alpha) < 0, x_N(\beta) > 0$ , and:

$$x_N(\beta) - x_N(\alpha) \geq 2. \tag{17}$$

Consider any  $n \in \mathbb{N}$  with  $n > N$ . We have either: (i)  $n \in M(\alpha)$  or (ii)  $n \in P(\alpha)$ . Case (ii) can be subdivided as follows: (a)  $n \in P(\alpha)$  and  $n \in P(\beta)$  and (b)  $n \in P(\alpha)$  and  $n \notin P(\beta)$ .

In case (i), we have  $n \in M(\alpha)$  and so  $n \in M(\beta)$ . However, because an additional element of  $M(\beta)$  has been used up for index  $N$ , compared with  $M(\alpha)$ , if  $n = m_k(\alpha)$ , we must have  $n = m_{k+j}(\beta)$  for some  $j \in \mathbb{N}$ . Therefore, by (15), we must have:

$$x_n(\beta) - x_n(\alpha) \geq 2. \tag{18}$$

In case (ii)(a), we have  $n \in P(\alpha)$  and  $n \in P(\beta)$ . However, because an additional element of  $P(\alpha)$  has been used up for index  $N$ , compared with  $P(\beta)$ , if  $n = p_r(\alpha)$ , we must have  $n = p_{r-j}(\beta)$  for some  $j \in \mathbb{N}$ . Thus, by (15), we must have:

$$x_n(\beta) - x_n(\alpha) \geq 2. \tag{19}$$

In case (ii)(b),  $n \in P(\alpha)$  and  $n \notin P(\beta)$ , so that  $n \in M(\beta)$ . That is,  $n \in L(\alpha, \beta)$ . Thus, by (15), we have  $x_n(\alpha) < 0, x_n(\beta) > 0$ , and:

$$x_n(\beta) - x_n(\alpha) \geq 2. \tag{20}$$

To summarize, for all  $n \geq N$ , we have:

$$x_n(\beta) - x_n(\alpha) \geq 2. \tag{21}$$

For  $n \in \mathbb{N}$  with  $n < N$  (if any), we have:

$$x_n(\beta) = x_n(\alpha). \tag{22}$$

**Step 3** (Comparing  $x(\alpha)$  with a finite permutation of  $x(\beta)$ )

Based on (21) and (22), we cannot say that  $W(\langle x_n(\alpha) \rangle) < W(\langle x_n(\beta) \rangle)$ , by invoking the Weak Pareto Axiom, except if  $N = 1$ , where (by (21)):

$$x_n(\beta) - x_n(\alpha) \geq 2 \text{ for all } n \in \mathbb{N}. \tag{23}$$

We consider now the case in which  $N > 1$ . We will show that (21) and (22) can be used to obtain:

$$x'_n(\beta) - x_n(\alpha) \geq 2 \text{ for all } n \in \mathbb{N},$$

where  $\langle x'(\beta) \rangle$  is a certain finite permutation of  $\langle x(\beta) \rangle$ .

Let  $n_1, \dots, n_{N-1}$  be the  $(N - 1)$  smallest elements of  $\mathbb{N}$  (with  $n_1 < \dots < n_{N-1}$ ) for which  $x_{n_i}(\alpha) < 0$  and  $x_{n_i}(\beta) > 0$  for  $i \in \{1, \dots, N - 1\}$ . Note that  $N = n_1$ . Then, define  $\langle x'_n(\beta) \rangle$  to be the sequence obtained by interchanging the  $i$ th entry of  $\langle x_n(\beta) \rangle$  with the  $n_i$ th entry of  $\langle x_n(\beta) \rangle$  for  $i = 1, \dots, N - 1$ , and leaving all other entries unchanged.

If  $i \in \{1, \dots, N - 1\}$  and  $x_i(\alpha) < 0$ , then:

$$x'_i(\beta) = x_{n_i}(\beta) > 0 \geq x_i(\alpha) + 2$$

and if  $x_i(\alpha) > 0$ , then by (16),

$$x'_i(\beta) = x_{n_i}(\beta) \geq x_i(\beta) + 2 \geq x_i(\alpha) + 2.$$

That is, in either case,

$$x'_i(\beta) \geq x_i(\alpha) + 2 \text{ for all } i \in \{1, \dots, N - 1\}. \tag{24}$$

If  $i \in \{1, \dots, N - 1\}$  and  $x_i(\beta) > 0$ , then because  $x_{n_i}(\alpha) < 0$ , we have:

$$x'_{n_i}(\beta) = x_i(\beta) > 0 \geq x_{n_i}(\alpha) + 2.$$

In addition, if  $x_i(\beta) < 0$ , then by (22),

$$x'_{n_i}(\beta) = x_i(\beta) = x_i(\alpha) \geq x_{n_i}(\alpha) + 2$$

using the fact that  $x_{n_i}(\alpha) < 0$  and  $i < n_i$ . That is, in either case,

$$x'_{n_i}(\beta) \geq x_{n_i}(\alpha) + 2 \text{ for all } i \in \{1, \dots, N - 1\}. \tag{25}$$

Finally, for  $n \in \mathbb{N}$  with  $n \notin \{1, \dots, N - 1\} \cup \{n_1, \dots, n_{N-1}\}$ , we have  $x'_n(\beta) = x_n(\beta)$ , and so by (21),

$$x'_n(\beta) \geq x_n(\alpha) + 2.$$

Thus, we have established that:

$$x'_n(\beta) \geq x_n(\alpha) + 2 > x_n(\alpha) + 1 \text{ for all } n \in \mathbb{N}.$$

Using the Anonymity and Weak Pareto Axioms, we have:

$$W(\langle x_n(\beta) \rangle) = W(\langle x'_n(\beta) \rangle) > W(\langle x_n(\alpha) + 1 \rangle). \tag{26}$$

**Step 4** (Non-overlapping intervals for distinct real numbers in  $(0,1)$ )

Define, for each  $t \in (0, 1)$ , a sequence  $\langle z_n(t) \rangle$  by:

$$z_n(t) = x_n(t) + 1 \text{ for all } n \in \mathbb{N}. \tag{27}$$

Note that the sequence  $\langle z_n(t) \rangle$  is in  $X$ , and by the Weak Pareto Axiom:

$$W(\langle z_n(t) \rangle) > W(\langle x_n(t) \rangle).$$

Therefore, for each  $t \in (0, 1)$ ,

$$I(t) = [W(\langle x_n(t) \rangle), W(\langle z_n(t) \rangle)] \tag{28}$$

is a non-degenerate closed interval in  $\mathbb{R}$ .

Let  $\alpha, \beta$  be arbitrary real numbers in  $(0, 1)$ , with  $\alpha < \beta$ . Then, by (26),

$$W(\langle x_n(\beta) \rangle) > W(\langle z_n(\alpha) \rangle). \tag{29}$$

Thus, the interval  $I(\beta)$  lies entirely to the right of the interval  $I(\alpha)$  on the real line.

That is, for arbitrary real numbers  $\alpha, \beta$  in  $(0, 1)$ , with  $\alpha \neq \beta$ , the intervals  $I(\alpha)$  and  $I(\beta)$  are disjoint. Thus, we have a one-to-one correspondence between the real numbers in  $(0, 1)$  (which is an uncountable set) and a set of non-degenerate, pairwise disjoint closed intervals of the real line (which is countable). This contradiction establishes the Proposition.  $\square$

We can now state the impossibility result for general domains of order type  $\mu$ .

**Proposition 5** *Let  $Y$  be a non-empty subset of  $\mathbb{R}$  such that  $Y(<)$  is of order type  $\mu$ . Then there is no social welfare function  $W : X \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms (where  $X = Y^{\mathbb{N}}$ ).*

PROOF: Suppose on the contrary that there is a social welfare function  $W : X \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms (where  $X = Y^{\mathbb{N}}$ ). Because  $Y(<)$  is of order type  $\mu$ ,  $Y$  contains a non-empty subset  $Y'$  such that  $Y'(<)$  is of order type  $\sigma$ . That is, there is a one-to-one mapping,  $g$ , from  $\mathbb{I}$  onto  $Y'$  such that:

$$a_1, a_2 \in \mathbb{I} \text{ and } a_1 < a_2 \implies g(a_1) < g(a_2).$$

Thus,  $g$  is an increasing function from  $\mathbb{I}$  to  $Y$ . Using Proposition 2, there is a function  $V : J \rightarrow \mathbb{R}$  satisfying the Weak Pareto and Anonymity Axioms, where  $J = \mathbb{I}^{\mathbb{N}}$ . However, this contradicts the result proved in Proposition 4, and establishes the result.  $\square$

We now discuss examples that illustrate the usefulness of Proposition 5.

**Example 4** Let  $Y = A \cup B$ , where  $A = \{-n/(1 + n)\}_{n \in \mathbb{N}}$  and  $B = \{n/(1 + n)\}_{n \in \mathbb{N}}$  and let  $X = Y^{\mathbb{N}}$ . Define  $f : \mathbb{I} \rightarrow \mathbb{R}$  by:

$$f(y) = \frac{y}{1 + |y|} \text{ for all } y \in \mathbb{I},$$

where  $\mathbb{I} \equiv \{n\}_{n \in \mathbb{N}} \cup \{-n\}_{n \in \mathbb{N}}$ . Then,  $f$  is an increasing function from  $\mathbb{I}$  onto  $Y$ . Therefore,  $Y(<)$  is similar to  $\mathbb{I}(<)$  and is of order type  $\sigma$ . By Proposition 5, there is no function  $W : X \rightarrow \mathbb{R}$  satisfying the Anonymity and Weak Pareto Axioms.

**Example 5** Let  $Y$  be the set of rationals in  $\mathbb{R}$ , and let  $X = Y^{\mathbb{N}}$ . Then, because  $\mathbb{I} \equiv \{n\}_{n \in \mathbb{N}} \cup \{-n\}_{n \in \mathbb{N}}$  is a subset of  $Y$ , and  $\mathbb{I}(<)$  is of order type  $\sigma$ ,  $Y(<)$  is of order type  $\mu$ . Thus, by Proposition 5, there is no function  $W : X \rightarrow \mathbb{R}$  satisfying the Anonymity and Weak Pareto Axioms.

**Example 6** Let  $Y$  be the set of positive rationals in  $\mathbb{R}$ , and let  $X = Y^{\mathbb{N}}$ . Define  $Y' = \{1/n\}_{n \in \mathbb{N}} \cup \{n\}_{n \in \mathbb{N}}$  and  $f : \mathbb{I} \rightarrow \mathbb{R}$  by:

$$f(y) = \begin{cases} y & \text{if } y \in B \\ 1/|y| & \text{if } y \in A \end{cases},$$

where  $\mathbb{I} \equiv A \cup B$ , with  $A = \{-n\}_{n \in \mathbb{N}}$  and  $B = \{n\}_{n \in \mathbb{N}}$ . Then,  $f$  is an increasing function from  $\mathbb{I}$  onto  $Y'$ . Thus,  $Y'(<)$  is similar to  $\mathbb{I}(<)$  and is therefore of order type  $\sigma$ . Because  $Y' \subset Y$ ,  $Y(<)$  is of order type  $\mu$ . By Proposition 5, there is no function  $W : X \rightarrow \mathbb{R}$  satisfying the Anonymity and Weak Pareto Axioms.

**Example 7** Let  $Y$  be the closed interval  $[0, 1]$  in  $\mathbb{R}$ , and let  $X = Y^{\mathbb{N}}$ . Define  $Z = \{1/n\}_{n \in \mathbb{N}} \cup \{n\}_{n \in \mathbb{N}}$ ,  $Y'$  to be the set of rationals in  $(0, 1)$ , and  $f : Z \rightarrow \mathbb{R}$  by:

$$f(y) = \frac{y}{1 + y} \text{ for all } y \in Z.$$

Then,  $f$  is an increasing function from  $Z$  into  $Y'$ . Thus,  $f(Z)(<)$  is similar to  $Z(<)$ , which is of type  $\sigma$  (by Example 6). Because  $Y'$  contains  $f(Z)$ ,  $Y'(<)$  is of type  $\mu$ . Because  $Y$  contains  $Y'$ ,  $Y$  contains  $f(Z)$ , and so  $Y(<)$  is of type  $\mu$ .

By Proposition 5, there is no function  $W : X \rightarrow \mathbb{R}$  satisfying the Anonymity and Weak Pareto Axioms. Our discussion of Example 7 provides an alternative proof for the impossibility theorem of Basu and Mitra (2007b, Theorem 4).



### 5 A reformulation of the main result

We have demonstrated that the complete characterization result in Theorem 1 can be applied to provide possibility and impossibility results for a variety of domains. Nevertheless, it will not have escaped the reader’s attention that checking the criterion involves checking all possible subsets of  $Y$  and determining whether any of these subsets is of the order type  $\sigma$ , the order type of the set of positive and negative integers. Checking whether a set in  $\mathbb{R}$  is of order type  $\sigma$  is relatively easy, given Lemma 1, but checking this for all possible subsets of  $Y$  might not be.

With this in mind, we devote this final section to a reformulation of the main result in terms of a criterion that involves looking at the *accumulation points* of  $Y$ .

For what follows,  $Y$  will be taken to be a non-empty subset of  $[0, 1]$ . If for an application one encounters a non-empty subset  $Y$  of  $\mathbb{R}$  that is not a subset of  $[0, 1]$ , one can always make a change of variable in the domain (through a monotone increasing function) so that the new domain  $Y'$  is a non-empty subset of  $[0, 1]$ . We have, in fact, done this already in discussing examples in Sections 3 and 4.

Because  $Y \subset [0, 1]$  is a subset of  $\mathbb{R}$ , it is possible to define *right accumulation points* and *left accumulation points* in the same spirit as right-hand limits and left-hand limits.

We will say that  $z \in \mathbb{R}$  is a *right accumulation point* of  $Y$  if given any  $\delta > 0$ , there is  $y \in Y$  such that:

$$0 < y - z < \delta.$$

Similarly, we will say that  $z \in \mathbb{R}$  is a *left accumulation point* of  $Y$  if given any  $\delta > 0$ , there is  $y \in Y$  such that:

$$0 < z - y < \delta.$$

Denote by  $R$  the set of right accumulation points of  $Y$  and by  $L$  the set of left accumulation points of  $Y$ . If  $Y$  has an infinite number of elements, then (being bounded) it will have an accumulation point.<sup>11</sup> Any accumulation point of  $Y$  will be either a right accumulation point or a left accumulation point or both. Furthermore, a right or left accumulation point of  $Y$  is also an accumulation point of  $Y$ .

Let us denote:

$$\rho \equiv \inf R \text{ and } \lambda \equiv \sup L$$

with the convention that if  $R$  is empty, then  $\rho = \infty$ , and if  $L$  is empty then  $\lambda = -\infty$ .

We can now state our characterization result as follows.

**Theorem 2** *Let  $Y$  be a non-empty subset of  $[0, 1]$ . There exists a social welfare function  $W : X \rightarrow \mathbb{R}$  (where  $X = Y^{\mathbb{N}}$ ) satisfying the Weak Pareto and Anonymity Axioms if and*

<sup>11</sup> For the standard definition of an accumulation point of a set, see Royden (1988, p. 46).

only if:

$$\rho \equiv \inf R \geq \sup L \equiv \lambda. \tag{30}$$

PROOF: (**Necessity**) Suppose condition (30) is violated; that is:

$$\inf R < \sup L. \tag{31}$$

Given the convention adopted, this means that  $\rho, \lambda$  are in  $\mathbb{R}$  and  $\rho < \lambda$ . It follows that there is a right accumulation point  $\rho'$  of  $Y$  and a left accumulation point  $\lambda'$  of  $Y$  such that  $\rho' < \lambda'$ .

Choose  $c \in (\rho', \lambda')$ . Then, we can find  $c < y_1 < y_2 < y_3 < \dots$  with  $y_k \in Y$  for  $k \in \mathbb{N}$  such that  $y_k \uparrow \lambda'$  as  $k \uparrow \infty$  (because  $\lambda'$  is a left accumulation point of  $Y$ ). Similarly, we can find  $c > y'_1 > y'_2 > y'_3 > \dots$  with  $y'_r \in Y$  for  $r \in \mathbb{N}$  such that  $y'_r \downarrow \rho'$  as  $r \uparrow \infty$  (because  $\rho'$  is a right accumulation point of  $Y$ ). That is, we have:

$$\rho' < \dots < y'_3 < y'_2 < y'_1 < c < y_1 < y_2 < y_3 < \dots < \lambda'. \tag{32}$$

Consider the set  $Y' = \{y_1 < y_2 < y_3 < \dots\} \cup \{y'_1 < y'_2 < y'_3 < \dots\}$ . Clearly,  $Y'$  is a subset of  $Y$  and because of (32), we note that (a)  $Y'$  has neither a maximum nor a minimum, and (b) for every cut  $[Y'_1, Y'_2]$  of  $Y'$ , the set  $Y'_1$  has a last element and the set  $Y'_2$  has a first element. Thus, by Lemma 1,  $Y'(<)$  is of order type  $\sigma$ . This means  $Y(<)$  is of order type  $\mu$ , and, by Theorem 1, there is no social welfare function  $W : X \rightarrow \mathbb{R}$  (where  $X = Y^{\mathbb{N}}$ ) satisfying the Weak Pareto and Anonymity Axioms.

(**Sufficiency**) Suppose (30) holds. We claim that  $Y(<)$  is not of order type  $\mu$ , for if it is of order type  $\mu$ , there is a non-empty subset  $Y' \subset Y$ , such that  $Y'(<)$  is of order type  $\sigma$ . Because  $Y' \subset [0, 1]$ , it has a greatest lower bound,  $a$ , and a least upper bound,  $b$ . Clearly,  $a \leq b$ .

Because  $Y'(<)$  is of order type  $\sigma$ , it does not have a maximum. Hence,  $b$  cannot be in  $Y'$ . Because  $b$  is a least upper bound of  $Y'$ , we can find  $y_1 < y_2 < y_3 < \dots$  with  $y_k \in Y'$  for  $k \in \mathbb{N}$  such that  $y_k \uparrow b$  as  $k \uparrow \infty$ . Then,  $b \in L$ , and so  $b \leq \rho$ .

Because  $Y'(<)$  is of order type  $\sigma$ , it does not have a minimum. Therefore,  $a$  cannot be in  $Y'$ . Because  $a$  is a greatest lower bound of  $Y'$ , we can find  $y'_1 > y'_2 > y'_3 > \dots$  with  $y'_r \in Y'$  for  $r \in \mathbb{N}$  such that  $y'_r \downarrow a$  as  $r \uparrow \infty$ . Then,  $a \in R$ , and so  $a \geq \rho$ .

Thus, we have:

$$a \geq \rho \geq \lambda \geq b \geq a,$$

so that  $a = b$ . However, then  $Y'$  must be a singleton and, therefore,  $Y'(<)$  cannot be of order type  $\sigma$ . This contradiction establishes our claim. Now, applying Theorem 1, there exists a social welfare function  $W : X \rightarrow \mathbb{R}$  (where  $X = Y^{\mathbb{N}}$ ) satisfying the Weak Pareto and Anonymity Axioms. □

**Remarks** We can now re-examine the examples in Sections 3 and 4 to see the applicability of Theorem 2 in deciding on possibility and impossibility results.

In the examples in Section 3, one can check that  $\rho \geq \lambda$ , so by Theorem 2 there exists a social welfare function  $W : X \rightarrow \mathbb{R}$  (where  $X = Y^{\mathbb{N}}$ ) satisfying the Weak Pareto and Anonymity Axioms.

In Example 1,  $Y = \mathbb{N}$ , which is similar to  $Y' = \{n/(1 + n)\}_{n \in \mathbb{N}}$ . Therefore, it is enough to examine  $Y'$ , which is a subset of  $[0, 1]$ . There is one left accumulation point (namely 1) and no right accumulation point. Hence,  $\rho = \infty$  while  $\lambda = 1$ , yielding  $\rho \geq \lambda$ .

In Example 2,  $Y = \{1/n\}_{n \in \mathbb{N}}$ , so there is one right accumulation point (namely 0) and no left accumulation point. Therefore,  $\rho = 0$ , while  $\lambda = -\infty$ , yielding  $\rho \geq \lambda$ .

In Example 3,  $Y = \{1/n\}_{n \in \mathbb{N}} \cup \{-1/n\}_{n \in \mathbb{N}}$ , so there is one right accumulation point (namely 0) and one left accumulation point (namely 0). Thus,  $\rho = 0 = \lambda$ .

In the examples in Section 4, one can check that  $\rho < \lambda$  (for a non-empty set  $Y'$  similar to  $Y$ ) so by Theorem 2 there does not exist any social welfare function  $W : X \rightarrow \mathbb{R}$  (where  $X = Y^{\mathbb{N}}$ ) satisfying the Weak Pareto and Anonymity Axioms.

In Example 4,  $Y = A \cup B$ , where  $A = \{-n/(1 + n)\}_{n \in \mathbb{N}}$  and  $B = \{n/(1 + n)\}_{n \in \mathbb{N}}$ . Then  $Y$  is similar to  $Y' = f(Y)$ , where  $f$  is given by:

$$f(y) = \left(\frac{1}{2}\right)(1 + y) \text{ for all } y \in \mathbb{Y}.$$

Then,  $Y'$  is a non-empty subset of  $[0, 1]$ . It has a right accumulation point at 0 and a left accumulation point at 1. Thus,  $\rho = 0 < 1 = \lambda$ .

In Example 5,  $Y$  is the set of rationals in  $\mathbb{R}$ . Then  $Y$  is similar to  $Y' = f(Y)$ , where  $f$  is given by (14). Then,  $Y'$  is a non-empty subset of  $[0, 1]$ , coinciding with the set of rationals in  $(0, 1)$ . Therefore, every point in  $[0, 1)$  is a right accumulation point, and every point in  $(0, 1]$  is a left accumulation point of  $Y'$ . Thus,  $\rho = 0 < 1 = \lambda$ .

In Example 6,  $Y$  is the set of positive rationals in  $\mathbb{R}$ . Then,  $Y$  is similar to  $Y' = f(Y)$ , where  $f$  is given by:

$$f(y) = \frac{y}{1 + y} \text{ for all } y \in \mathbb{Y}.$$

Then,  $Y'$  coincides with the set of rationals in  $(0, 1)$ , and (as in Example 5), we have  $\rho < \lambda$ .

In Example 7,  $Y = [0, 1]$ . Then, every point in  $[0, 1)$  is a right accumulation point, and every point in  $(0, 1]$  is a left accumulation point of  $Y$ . Thus,  $\rho = 0 < 1 = \lambda$ .

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